Application for Funding to the College Academy for Research, Scholarship, and Creative Activity (CARSCA)- Mathematics and Natural Sciences

February 25, 2013

1. **Project Title**

When are two operators "the same" in a mathematical sense?
The search for a criteria for unitary equivalence and similarity

2. **PI**

Hyun-Kyoung Kwon, Ph.D., Assistant Professor, Department of Mathematics

3. **Abstract**

One of the most fundamental problems in mathematics is to determine whether two objects are identical. An undergraduate student taking a course in linear algebra learns that in mathematical parlance, unitary equivalence or similarity of two matrices means that these matrices are "the same". For matrices, this problem is completely solved and one asks the same question when the objects of interest are now operators. Although matrices and operators have something in common in the sense that they both represent mathematical relations, the situation becomes much more complicated when operators enter the picture. There are several partial results known in the research field but a general criteria is nonexistent. In the proposed project, the PI takes the approach introduced by M. J. Cowen and R. G. Douglas in the late 1970's of using the eigenvectors of the operators and combines this viewpoint with results from function theory to formulate a comprehensive criteria as expansive as possible that will include all previous partial results. The outcome of the project will not only contribute to the formulation of a general criteria for operator equivalence but also facilitate the exchange of ideas and methods among different research areas of mathematics due to its interdisciplinary character. It will also be a basis for grant proposals that the PI plans on submitting to the National Science Foundation and to the Simons Foundation during the following academic year. Lastly, the PI will try to get a gradute student interested in the current research problem involved in the project.

Signatures:

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Hyun-Kyoung Kwon, PI

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Zhijian Wu, Chair of the Department of Mathematics

4. **Proposal**

(1) **Aim of the project**

One easily finds in undergraduate linear algebra texts that Specht's Theorem and the comparing of the Jordan canonical forms provide ways of completely deciding whether two arbitrary matrices are identical in a mathematical sense- unitarily equivalent or similar. Many more challenging complications arise when we deal with operators (infinite-dimensional analogues of matrices), and there exist no such general criteria. Only a few partial results are known to this date ([1], [3], [4], [5], [8], [11], [12], [14]) and these results themselves entailed the formulation of novel theories in the research field or an ingenious application of the results and methods of another research field adapted to the operator setting.

The purpose of the proposed project consists in generalizing the results previously obtained by the PI through collaborations with other mathematicians of the field ([3], [4], [5], [8]) to the point that will enable her to make a conjecture and develop a general criteria pertaining to operators. Such criteria will be consistent with the partial results mentioned above and it will play the same role for operators as Specht's Theorem and the Jordan canonical form do for matrices.

In their pioneering work of the late 1970's, M. J. Cowen and R. G. Douglas [4] introduced the idea of using the eigenvectors of operators to solve this problem in a special case, taking a geometric viewpoint. The PI's past and current research has been concentrated on this standpoint and will continue to be so in the proposed project. It is worth noting that although there are significant differences in working with operators as opposed to dealing with matrices, the problem heavily relies on the concepts of eigenvalues and eigenvectors familiar to a linear algebra student.

Moreover, different research areas of mathematics including operator theory, function theory, complex differential geometry, harmonic analysis, Hilbert module theory, and differential equations have provided ideas and techniques used in the PI's previous work and they will continue to make the current project take on an interdisciplinary

shape. As an example, the corona theorem of function theory lets one transform this equivalence problem to the corona problem. Also, the conjecture is that a criteria should be related to the exxistence of a solution to a certain differential equation and this solution is based on an important concept in harmonic analysis. In mathematics, it is important to discover the relationship between two seemingly different problems and this project demonstates this relationship in several ways. Another aim of the proposed project is to remind mathematicians of the importance of an interdisciplinary approach and encourage them to take a broad view of the problem they are working on rather than to stick to one specific methodology.

(2) **Proposed expenditure**

The PI, who just started her tenure-track position at the University of Alabama, does not have external grants available at the moment that can be used for travel. In the PI's research field, conferences provide opportunities to not only present new results but also to meet with other mathematicians for exchange of ideas and agreeing on collaboration. In fact, the fruitful collaborations with R. G. Douglas began at a conference where the PI presented her thesis result and through subsequent visits to him, she has been able to closely work with him for the past several years.

The PI has applied for a one-time SEC travel grant and an RGC grant to make visits to her collaborators. The PI is currently waiting for their decision but these grants do not provide support for conference participations which is also an integral part of carrying out research. Without the CARSCA funds, the PI will need to ask for partial travel support from the organizers of the conferences she hopes to attend and some of them might not even provide such support.

(3) **Future funding**

The PI plans on applying for external grants in the following academic year to the National Science Foundation and to the Simons Foundation. By carrying out the

proposed research, the PI will continue to develop into an experienced mathematician and will be able to come up with new ideas and techniques about the research problem. Currently the PI considers four different directions for generalizations but during the course of the project, she might be able to come up with other directions as well, thus getting closer and closer to the final goal of identifying two operators. Moreover, the outcomes of the project will be used to better design the proposals that will be submitted to various foundations, resulting in stronger proposals containing more definite project plans and conjectures.

(4) **Enhancement of proposed activity**

The PI's past results on the proposed problem attracted attention among many established mathematicians in the field. In fact, the result of [8] provided the first partial result to the question asked by M. J. Cowen and R. G. Douglas in the late 1970's. It is expected that such an interest could be aroused again through the outcome of the project and thus contribute to the development of the PI's research field. The PI will gain more research experience and will mature into an established mathematician. Moreover, the PI will be offered the opportunity to have conversations with other mathematicians who are her prospective collaborators. Lastly, the project might be able to provide the PI the opportunity of supervising a student and guiding him or her through research for the very first time in her life.

(5) **Evaluation for success**

The outcomes of the proposed project will be made explicit through published papers in quality journals and through conference and seminar talks, both domestic and international. The success will also be measured in part by the grant proposal submissions to the National Science Foundation in October of 2013 and to the Simons Foundation in January of 2014.

(6) **Budget**

Domestic: $ 2,000 for PI to attend 1-3 conferences, usually to JMM, SEAM, and Texas A & M Workshop on Analysis and Probability.

International: $ 3,000 for PI to attend one conference, usually to IWOTA.

* The figures are based on the amount requested in the application for the DMS grant from the National Science Foundation.

Timeline

(a) summer of 2013: investigation of the ideas that will be used in the project and the checking of their validity

(b) fall of 2013- spring 2014: carrying out of the research according to the PI's plan

(c) summer of 2014: paper submissions

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